Actuarial Geometry

Stephen Mildenhall
Aon Re Services, Chicago IL

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Outline

1. Framework
2. Losses are not homogeneous with respect to volume
3. Insurance risk is not volumetrically diversifying
4. Homogeneous model is not even "locally" appropriate
5. Empirical data and supporting evidence
6. Four models based on Levy processes
7. Why bother with general Levy processes vs. compound Poisson processes?
8. So what? Can we see impact in prices?

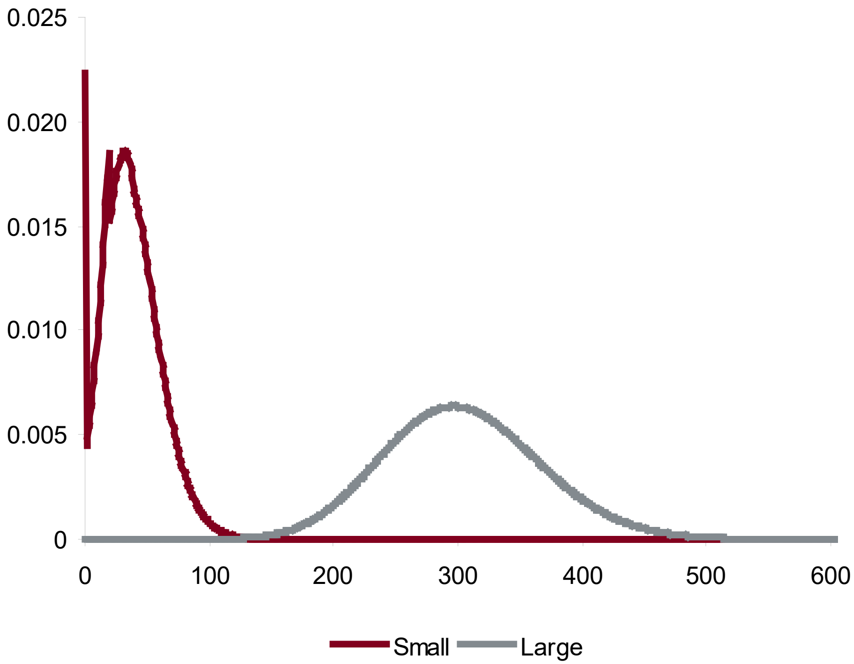
1. Framework

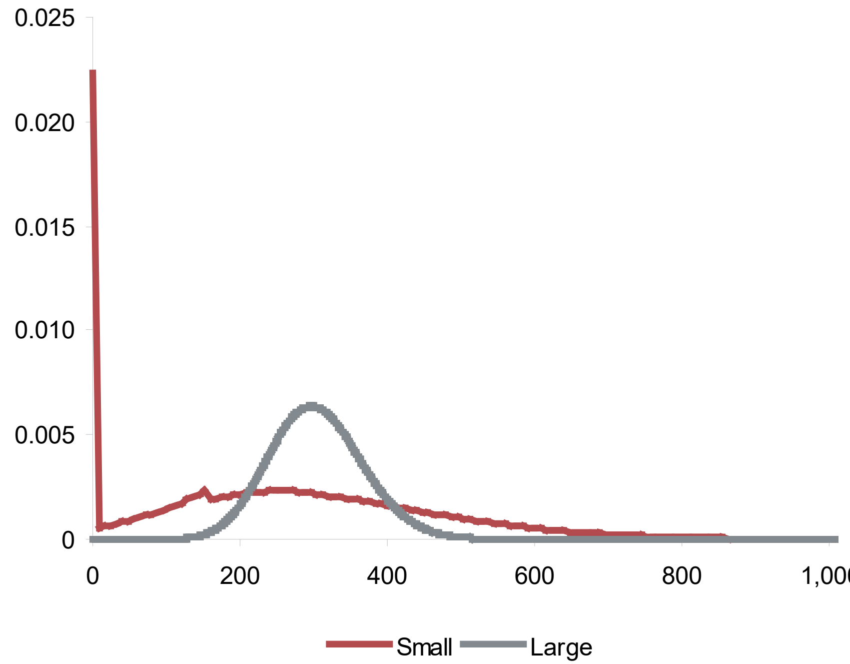
	Risk Theory	Finance	Actuarial
1900s	Bachelier		Bureau rates
	Lundberg		Bureau rates
1930s	Levy, Kolmogorov, Khintchine		Bureau rates
1950s			…
		Portfolio theory	Bureau rates
1960s		CAPM	Investment income
	Buhlmann Borch	Systemic vs. diversifiable risk	Ferrari, ROE
1970s			1978 u/w profit
		Option pricing	
1980s		Fair rate of return	
1990s	Artzner et al. Coherent Measure of Risk		Cat Models
		Phillips, Cummins	
2000s	Convex risk measures		2004 u/w profit
		Froot et al. Merton-Perold Zanjani Boyer	Idiosyncratic risk matters

2. Losses are not Homogeneous wrt Volume

- Expected Loss ($) = Volume ($ / t) x Time (t)
  - For fixed t (t=1), expected loss = volume
- A(x,t) := aggregate losses from volume x insured for time t
  - E[A(x,t)] = xt = expected loss
- **Homogeneous** model: $A(x,t) = xR_t$
  - R_t a “return” variable
  - For assets: x is position size and R_t is return or unit price
- Homogeneity implies
  - Shape of aggregate loss distribution independent of volume
  - No volume based diversification
  - A(x,t) has constant coefficient of variation (volatility) with x
  - Recall **coefficient of variation** = CV = standard deviation / mean
- Homogeneous models are not appropriate for insurance
  - Consider probability of zero losses: Pr(xX=0)=Pr(X=0) independent of volume x

2. Losses are not Homogeneous wrt Volume





- Consider probability of zero claims in small and large books
- Compound Poisson aggregate losses
  - Small: claim count 4
  - Large: claim count 32
- Left plot unscaled; right plot scaled
- Homogeneous distributions would be indistinguishable in scaled plot
  - Note decrease in variance on right hand plot

2. Losses are not Homogeneous wrt Volume

- Geometric Brownian motion model is homogeneous wrt volume
  - $S_t = S_0 \exp((\mu - \sigma^2/2) t + \sigma B_t)$, where B_t is a Brownian motion
  - Volume = S_0
  - Return = $\exp((\mu - \sigma^2/2) t + \sigma B_t)$

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3. Insurance Risk is not Volumetrically Diversifying

- Meaning
  - CV(A(x,t)) does not tend to zero as x increases, for fixed t

- Practical meaning
  - It is impossible to diversify away all insurance risk by growing larger

- How to investigate?
  - CV(A) = CV(A / p) = CV(loss ratio), p = fixed premium
  - Look at volatility in loss ratio with volume

- Data source: NAIC Annual Statement, Schedule P
  - Gross, ultimate loss ratios
  - 10 accident year history
  - Major lines: WC, Commercial Auto, HO, PPA, CMP, Other Liability etc.

SCHEDULE P - PART 1D - WORKERS' COMPENSATION

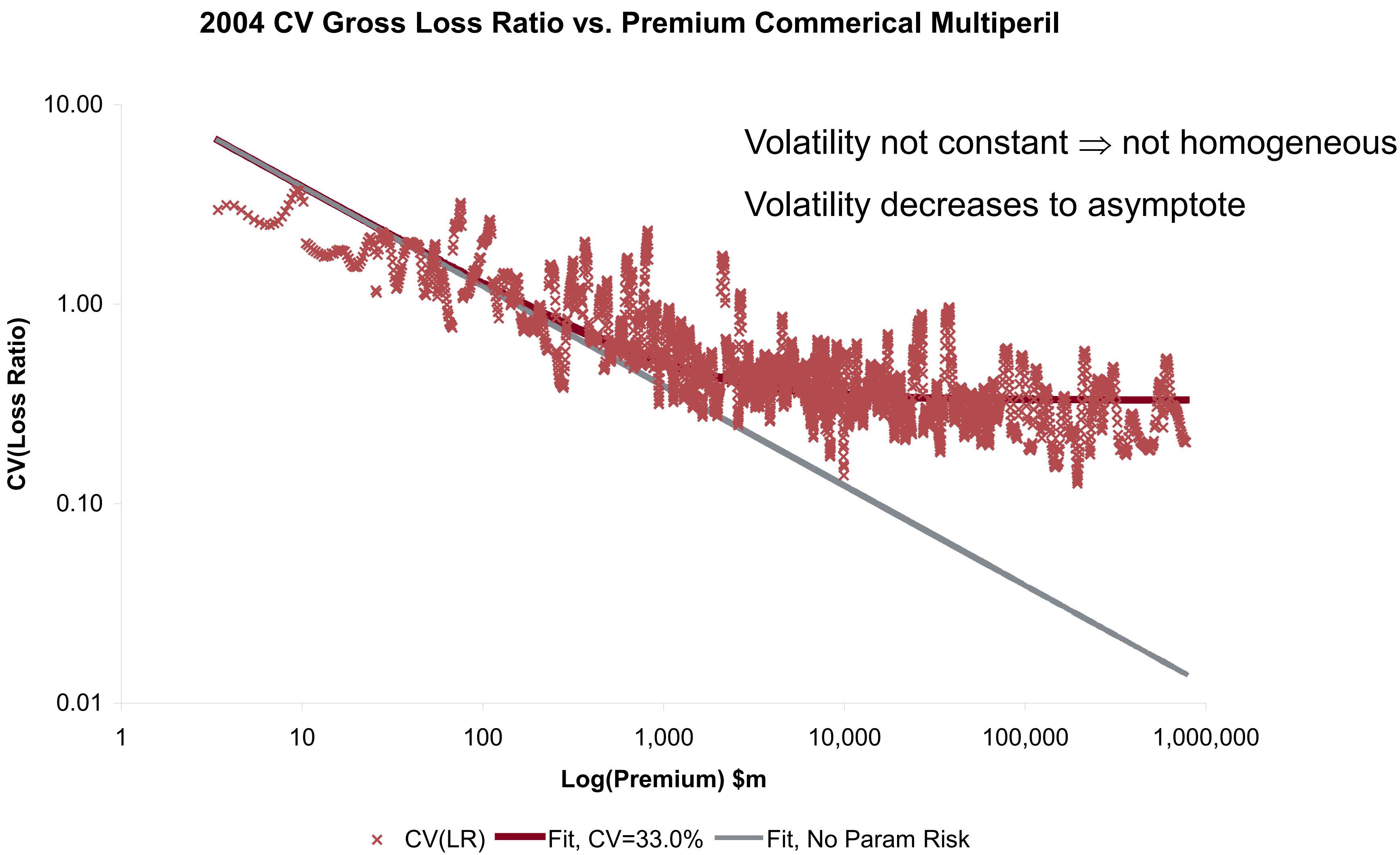
($000 omitted)

Years in Which Premiums Were Earned and Losses Were Incurred	Premiums Earned			Loss and Loss Expense Payments								12
				Loss Payments		Defense and Cost Containment Payments		Adjusting and Other Payments		10	11	Number of Claims Reported-Direct and Assumed
	1 Direct and Assumed	2 Ceded	3 Net (Cols. 1 - 2)	4 Direct and Assumed	5 Ceded	6 Direct and Assumed	7 Ceded	8 Direct and Assumed	9 Ceded	Salvage and Subrogation Received	Total Net Paid (Cols. 4 - 5 + 6 - 7 + 8 - 9)	
1. Prior	XXX	XXX	XXX	156,422	7,531	10,365	72	3,530	(10)	33,847	162,724	XXX
2. 1996	1,746,768	9,914	1,736,854	859,568	12,096	66,293	1,254	93,376	8	51,572	1,005,879	348,154
3. 1997	1,342,521	(151,161)	1,493,682	1,012,510	11,854	83,723	1,705	73,653	0	60,094	1,156,327	384,917
4. 1998	1,704,209	28,043	1,676,166	1,303,449	(17,438)	110,787	2,411	94,356	5	66,457	1,523,614	423,447
5. 1999	1,723,216	270,103	1,453,113	1,409,971	413,039	115,987	12,402	81,396	9	64,051	1,181,904	424,836
6. 2000	1,390,797	194,283	1,196,514	1,104,815	412,884	103,020	14,787	45,466	4	54,749	825,625	357,680
7. 2001	1,037,840	583,732	454,108	888,300	411,142	80,207	13,631	65,486	24	39,772	609,196	292,642
8. 2002	1,464,414	180,605	1,283,809	583,945	47,368	56,075	2,123	83,137	0	27,115	673,665	226,035
9. 2003	1,517,227	426,236	1,090,991	436,436	44,729	40,689	1,652	63,550	0	12,505	494,294	158,810
10. 2004	1,504,575	208,397	1,296,178	274,586	32,821	24,354	1,159	34,879	0	4,999	299,839	131,659
11. 2005	1,173,428	205,268	968,160	89,976	4,481	5,991	201	29,498	(115)	505	120,898	82,681
12. Totals	XXX	XXX	XXX	8,119,978	1,380,507	697,490	51,398	668,328	(75)	415,666	8,053,965	XXX

	Losses Unpaid				Defense and Cost Containment Unpaid				Adjusting and Other Unpaid		23	24	25
	Case Basis		Bulk + IBNR		Case Basis		Bulk + IBNR						
	13 Direct and Assumed	14 Ceded	15 Direct and Assumed	16 Ceded	17 Direct and Assumed	18 Ceded	19 Direct and Assumed	20 Ceded	21 Direct and Assumed	22 Ceded	Salvage and Subrogation Anticipated	Total Net Losses and Expenses Unpaid	Number of Claims Outstanding-Direct and Assumed
1. Prior	1,466,509	101,202	386,407	171,778	0	0	61,370	124	47,577	(411)	41,783	1,689,170	11,556
2. 1996	107,654	1,197	13,695	(100)	0	0	11,487	20	4,376	0	868	136,095	895
3. 1997	157,495	1,506	24,545	2,825	0	0	20,896	20	4,728	0	1,225	203,313	1,204
4. 1998	230,749	3,959	42,455	83	0	0	18,732	118	6,869	(141)	2,057	294,786	1,939
5. 1999	297,512	78,852	36,042	13,571	0	0	7,161	144	6,207	0	3,303	254,355	2,833
6. 2000	275,517	99,191	58,059	27,994	0	0	23,111	465	6,721	0	1,229	235,758	3,172
7. 2001	289,330	424,745	124,500	84,165	0	0	20,449	492	6,548	0	151	(68,575)	3,067
8. 2002	188,949	16,288	130,920	20,025	0	0	37,282	926	8,618	0	16,653	328,530	2,494
9. 2003	181,644	24,726	171,086	18,836	0	0	36,611	904	9,118	0	16,486	353,993	3,095
10. 2004	181,275	24,505	320,179	71,518	0	0	45,246	3,162	11,366	0	20,379	458,881	4,783
11. 2005	147,550	8,236	402,348	85,228	0	0	49,821	2,236	71,660	0	21,843	575,679	13,435
12. Totals	3,524,184	784,407	1,710,236	495,923	0	0	332,166	8,611	183,788	(552)	125,977	4,461,985	48,473

	Total Losses and Loss Expenses Incurred			Loss and Loss Expense Percentage (Incurred/Premiums Earned)			Nontabular Discount		34	Net Balance Sheet Reserves after Discount	
	26 Direct and Assumed	27 Ceded	28 Net	29 Direct and Assumed	30 Ceded	31 Net	32 Loss	33 Loss Expense	Inter-Company Pooling Participation Percentage	35 Losses Unpaid	36 Loss Expenses Unpaid
1. Prior	XXX	XXX	XXX	XXX	XXX	XXX	0	0	XXX	1,579,936	109,234
2. 1996	1,156,449	14,475	1,141,974	66.2	146.0	65.7	0	0	0.00	120,252	15,843
3. 1997	1,377,550	17,910	1,359,640	102.6	(11.8)	91.0	0	0	0.00	177,709	25,604
4. 1998	1,807,397	(11,003)	1,818,400	106.1	(39.2)	108.5	0	0	0.00	269,162	25,624
5. 1999	1,954,276	518,017	1,436,259	113.4	191.8	98.8	0	0	0.00	241,131	13,224
6. 2000	1,616,709	555,325	1,061,383	116.2	285.8	88.7	0	0	0.00	206,391	29,367
7. 2001	1,474,820	934,199	540,621	142.1	160.0	119.1	0	0	0.00	(95,080)	26,505
8. 2002	1,088,925	86,730	1,002,195	74.4	48.0	78.1	0	0	0.00	283,556	44,974
9. 2003	939,134	90,847	848,287	61.9	21.3	77.8	0	0	0.00	309,168	44,825
10. 2004	891,885	133,165	758,720	59.3	63.9	58.5	0	0	0.00	405,431	53,450
11. 2005	796,844	100,267	696,577	67.9	48.8	71.9	0	0	0.00	456,434	119,245
12. Totals	XXX	XXX	XXX	XXX	XXX	XXX	0	0	XXX	3,954,090	507,895

3. Risk is not Volumetrically Diversifying



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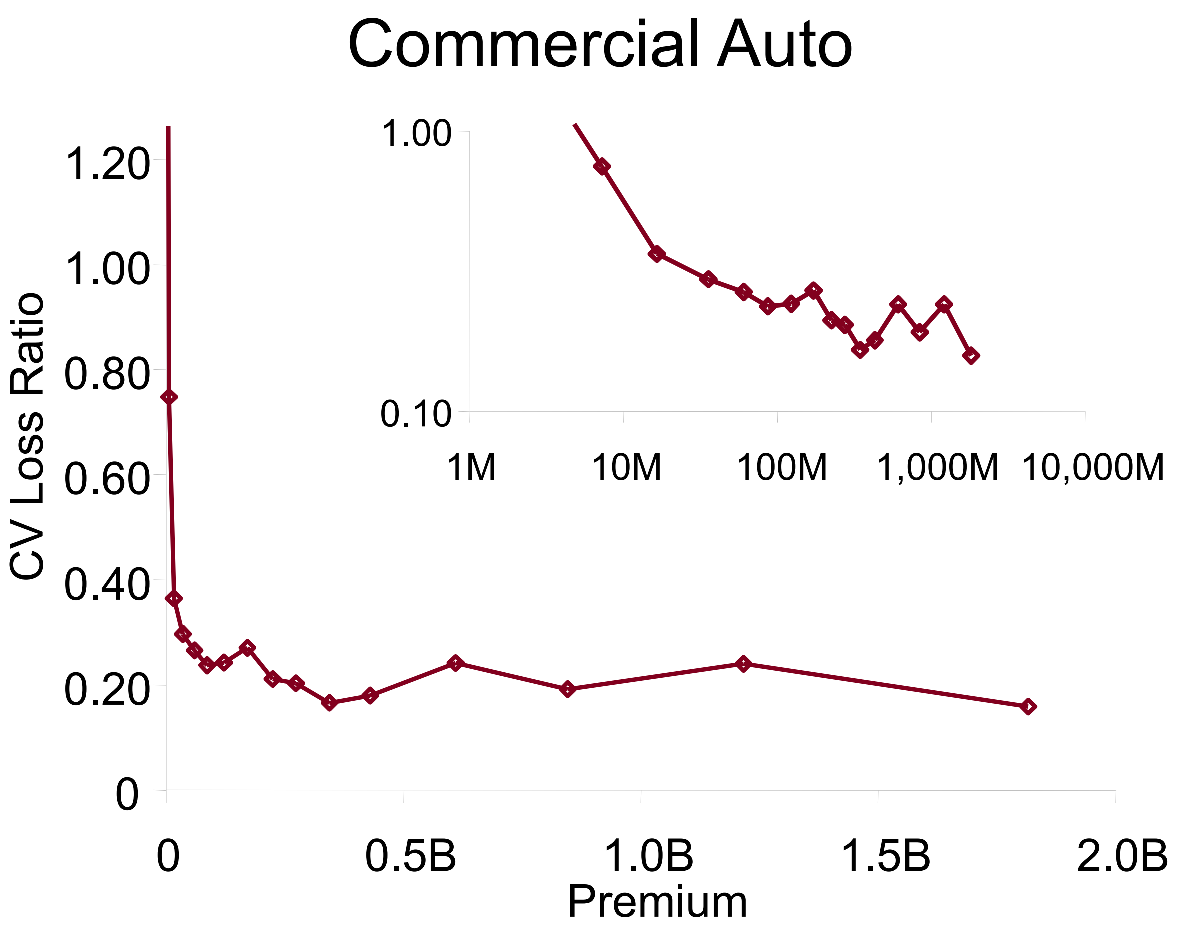

Risk not constant – not homogeneous

Risk decreases to asymptote

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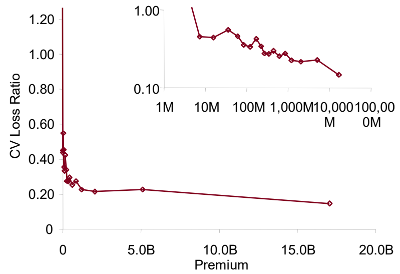
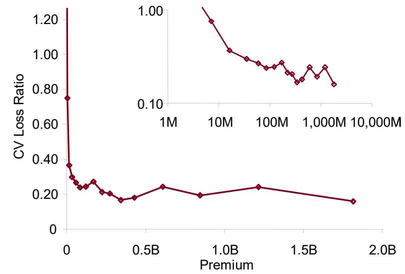
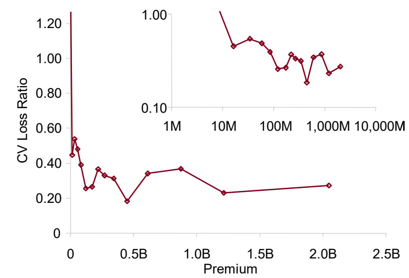
All Lines



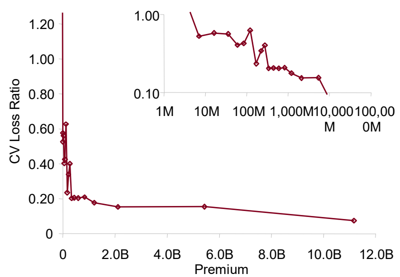
Commercial Auto



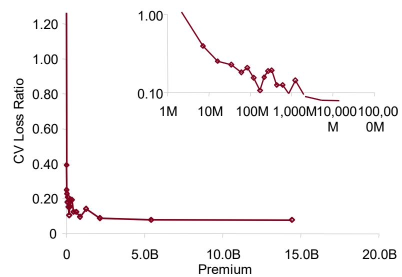
Commercial Multiperil



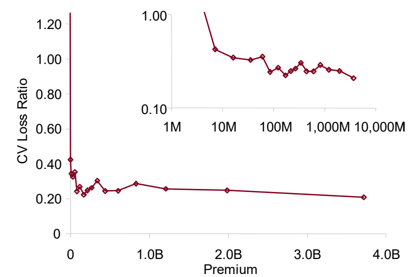
Homeowners



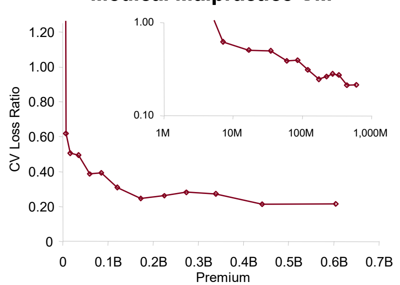
Private Passenger Auto



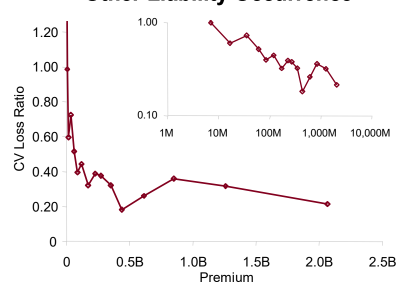
Workers Compensation



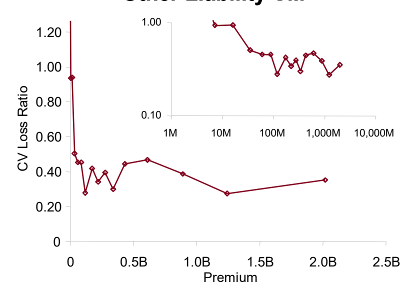
Medical Malpractice CM



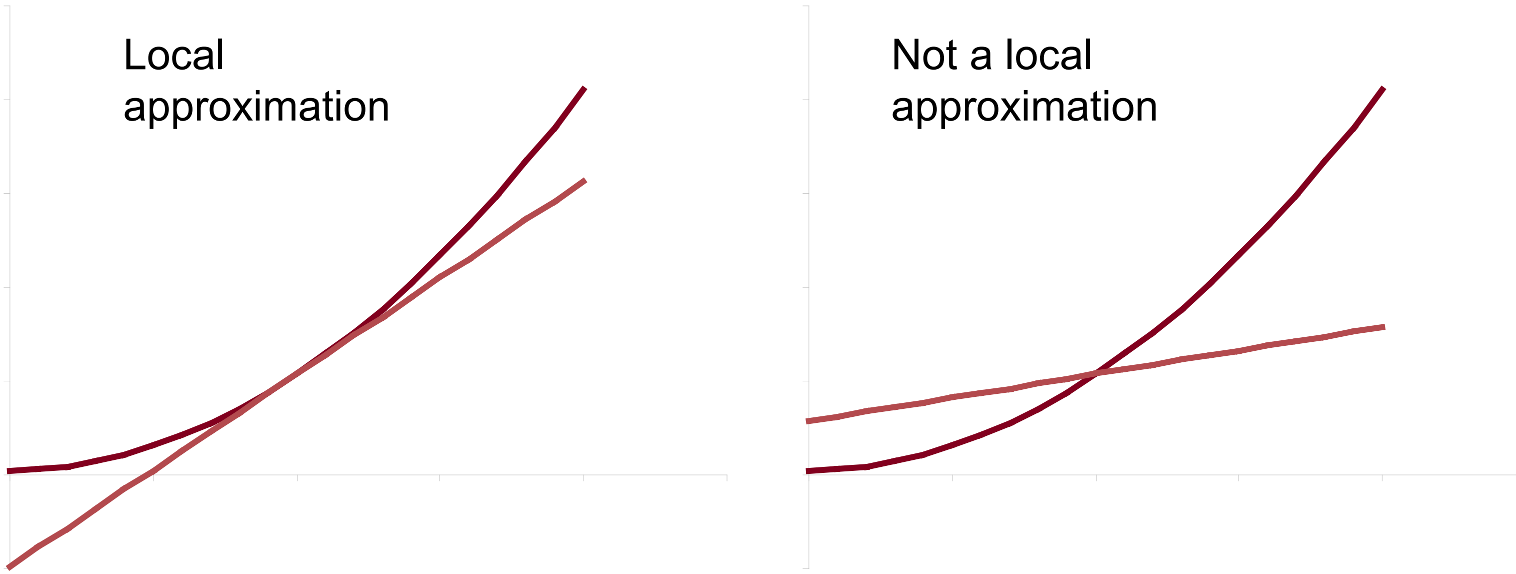
Other Liability Occurrence



Other Liability CM



4. Homogeneity is not "locally" appropriate



- Local approximation: one holding to first order in a neighborhood of a point
  - First-order equality required by any theory considering derivatives (Myers-Read)
  - Equality ≠ first order approximation

- Requires notion of **derivative** which requires a **direction**

4. Homogeneity is not "locally" appropriate

- For simplicity, ignore volume x, and assume x=1
- Model X(t) as a (mixed) compound Poisson distribution = family aggregate losses and suppose
  - Expected claim count = t
  - E(severity) = 1
  - So E(X(t)) = E(severity) x E(claim count) = t
- A homogeneous approximation to the family X(t) near t=1 is given by t X(1)
  - Will show this is **not** a local approximation
- Have two maps from $[0,\infty) \rightarrow$ { risks }, agreeing at t=1:
  - m(t) = X(t), Meyers embedding
  - k(t) = t X(1), asset or Kalkbrener embedding

4. Homogeneity is not "locally" appropriate

- Let ρ : { risks } → **R** be a real-valued risk measure
  - Standard deviation, downside risk, higher moment, percentile (=Value-at-Risk, VaR), TVaR
  - Tasche, Denault, Fischer, Myers-Read,… show we should be interested in $\partial \rho / \partial t$, the rate of change of ρ with volume in the line

- Two compositions $\rho \circ k$, $\rho \circ m$: $[0,\infty) \to$ **R** both give single valued functions of a single real variable

- Meyers, RTS 2005, showed for ρ = standard deviation
  - $\partial (\rho \circ k) / \partial t \neq \partial (\rho \circ m) / \partial t$

- In terms of derivatives of ρ (picture):
  - $\partial \rho \circ k / \partial t = D\rho_{X(1)}(k'(1))$ and $\partial \rho \circ m / \partial t = D\rho_{X(1)}(m'(1))$
  - Implies **directions m'(1) ≠ k'(1)**

- What are the directions m'(1) and k'(1)?

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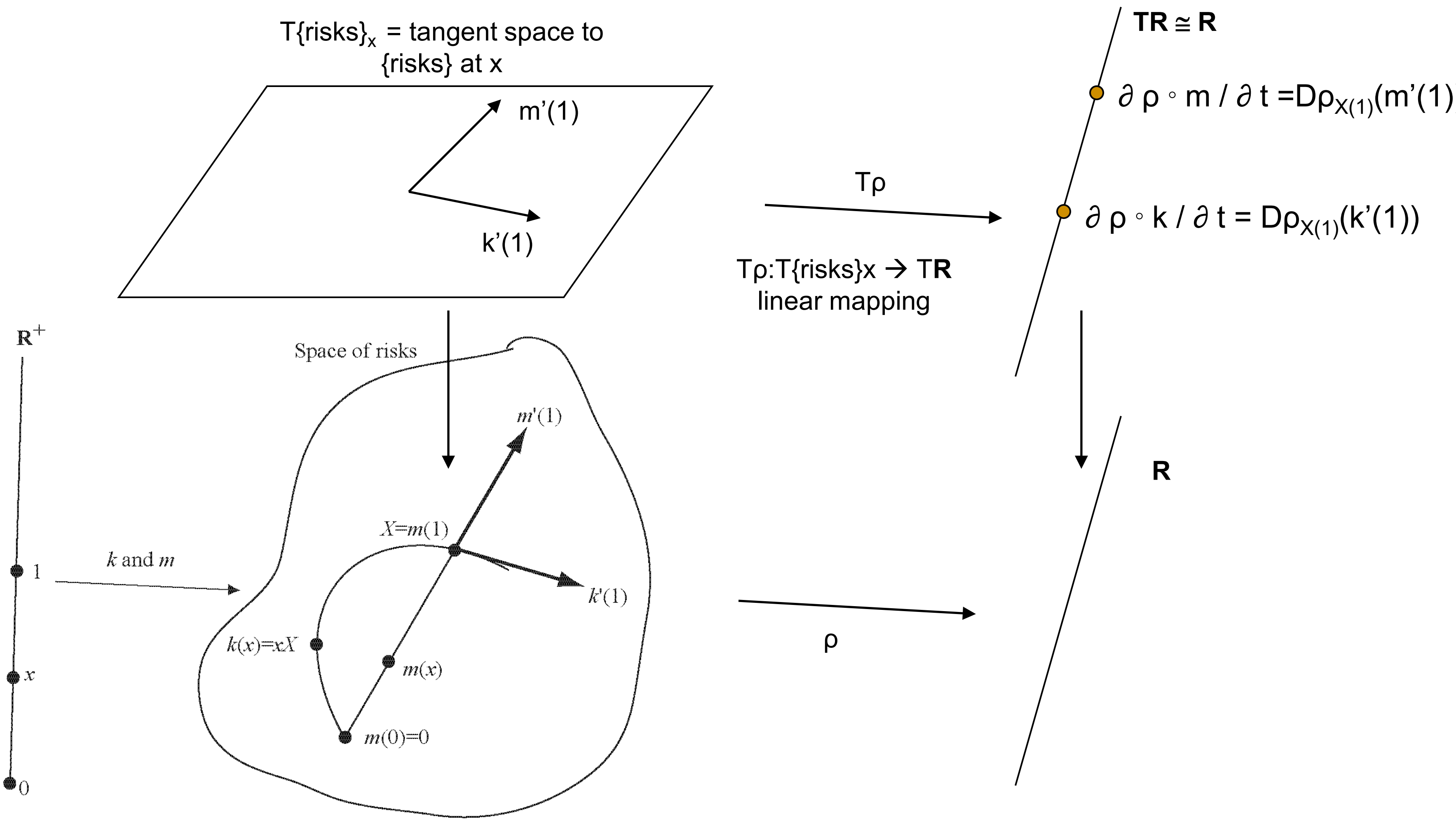
4. Homogeneity is not "locally" appropriate



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4. Homogeneity is not "locally" appropriate

- Why is m drawn as the straight line?

Table 1: Possible characterizations of a ray in $\mathbb{R}^n$

Characterization of ray	Required structure on $\mathbb{R}^n$
α is the shortest distance between $\alpha(0)$ and $\alpha(1)$	Notion of distance in $\mathbb{R}^n$, differentiable manifold
$\alpha''(t) = 0$, constant velocity, no acceleration	Very complicated on a general manifold.
$\alpha(t) = t\mathbf{x}$, $\mathbf{x} \in \mathbb{R}^n$.	Vector space structure
$\alpha(s+t) = \alpha(s) + \alpha(t)$	Can add in domain and range, semi-group structure only.

- What is the addition operator "+" in { risks }?
  - Assets: vector space structure with basis of return variables (3X ok)
  - Insurance: convolution of random variables (3X not ok, $X_1 + X_2 + X_3$)

4. Homogeneity is not "locally" appropriate

- Defining property for straight-line in { risks }
  - $m(s + t) = m(s) \star m(t)$, convolution sum of random variables

- Levy process satisfies $m(s + t) = m(s) \star m(t)$
  - Additive, independent, homogeneous increments, stochastically continuous

- Examples of Levy processes
  - Brownian motion, compound Poisson, drift, combinations

- What are k'(1) and m'(1)?
  - m_t defines a family of probability measures
  - Properties manifest through **operator action** on functions $\langle f,m_t \rangle = \int f(x)\, dm_t(x)$
  - Derivative should be a **family of linear functionals** $f \rightarrow m'(f)(t)$ indexed by t
  - Fundamental Theorem of Calculus: $\langle f,m(1) \rangle - \langle f,m(0) \rangle = \int m'(f)(t)\, dt$
  - Differentially: $m'(f)(0) = \lim_{t \rightarrow 0+} [\, E(f(X_t) - f(X_0))\,] / t$, where X_t has distribution m(t)

4. Homogeneity is not "locally" appropriate

- $\lim_{t\to 0}$ [E(f(X_t) – f(X_0))] / t defines infinitesimal generator of Markov process
- For compound Poisson m, let J be distribution of jump sizes, E(J)=1
- For small t, Pr(jump) = λt, so, conditioning on presence of a jump
  - E(f(X_t)) = λt E(f(J)) + (1 - λt) f(0)

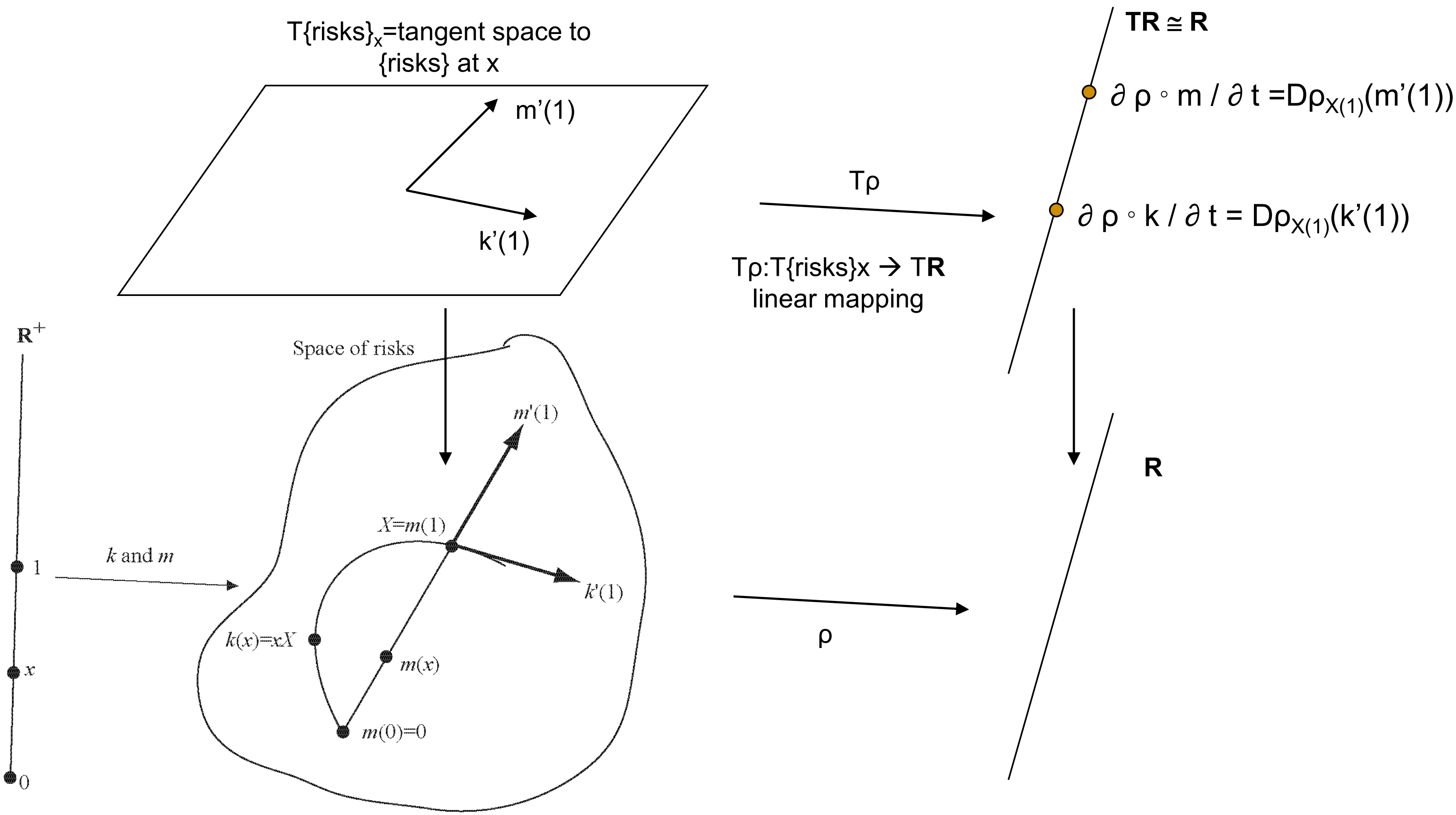
  and hence
  - m'(f)(0) = λ (E(f(J)) – f(0))
- For k, E(f(X_t)) = E(f(tX)) = f(0) + tE(X) f'(0) + O(t^2), so
  - k'(f)(0) = E(X) f'(0), which is *completely different*

Example
J=1, constant, X=Poisson(λ).

m'(f)(0) = λ (f(1)) – f(0))

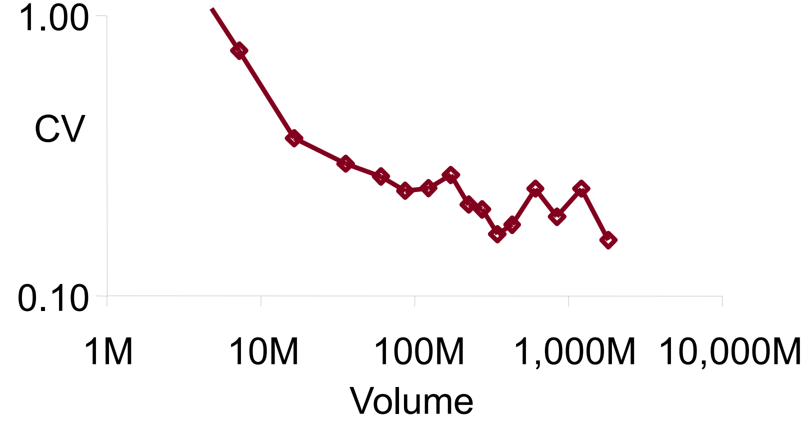
k'(f)(0) = E(X) f'(0) = λ f'(1)

4. Homogeneity is not "locally" appropriate



5. Empirical Evidence

- Data supports two hypotheses
  - Risk is not homogeneous: CV not constant wrt volume
  - Risk is not volumetrically diversifying: CV has asymptote



- Can we say more?

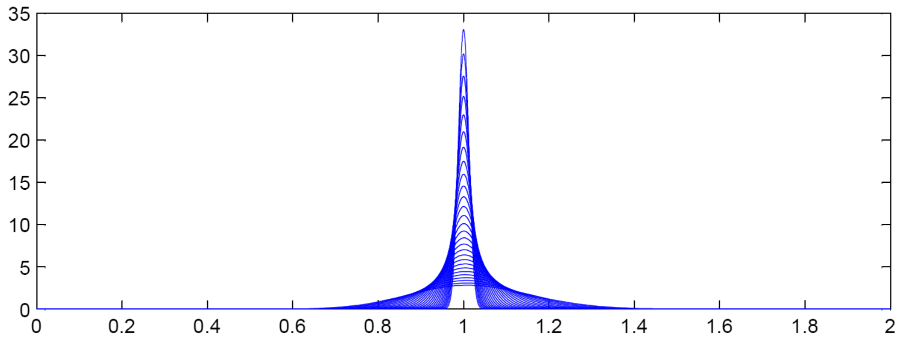
- Levy process based models
  - A(x,t) = X(xt)
  - A(x,t) = X(xZ(t)), Z a positive, increasing Levy process (a subordinator)
  - A(x,t) = X(xCt), E(C)=1, C is called a **mixing variable**
  - A(x,t) = X(xCZ(t))

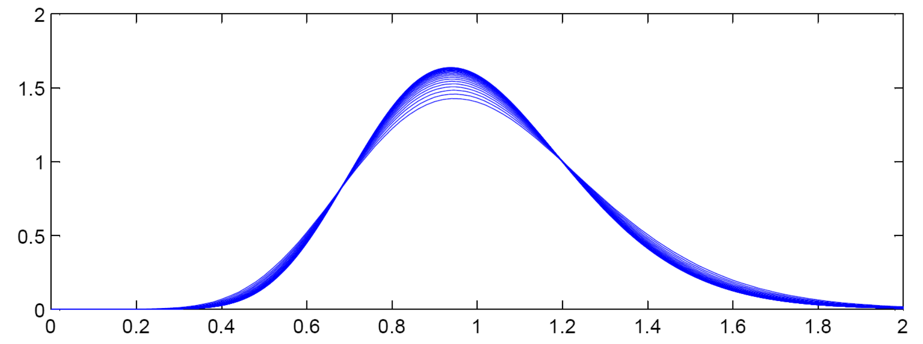
Mixing Variables & the Distribution of Normalized Loss Ratios

- Mixed compound Poisson: $A = X_1 + \ldots + X_N$, $N|C \sim \text{Poisson}(nC)$, $E(C)=1$
- Normalized Loss Ratio NLR = A / E(A)
- Dichotomous behavior of normalized loss ratios

No parameter uncertainty: leads to unrealistic aggregate loss distribution as expected losses increase

Including parameter preserves actual variability observed in data for large insurers





If C is constant, NLR converges to 1.0 in distribution

Illustration shows aggregates with Poisson frequency & larger & larger values of E(A)

If C is not constant, NLR converges to C in distribution

Illustration shows aggregates with negative binomial frequency (gamma mixing) & larger & larger values of E(A)

Key Technical Result

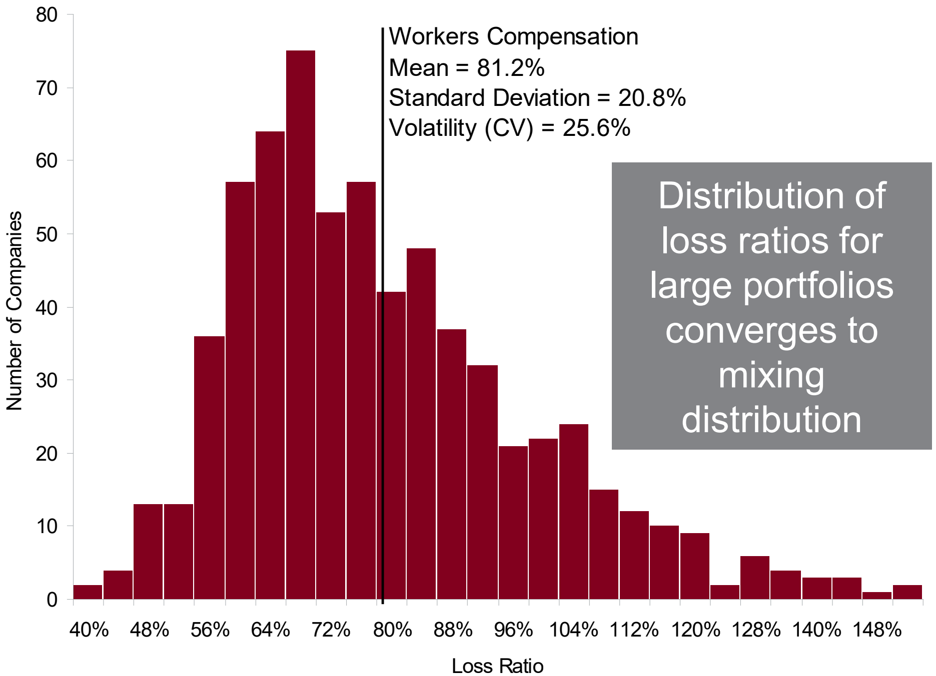
- If severity X has a variance then A / E(A) converges in distribution to C
- Proof:

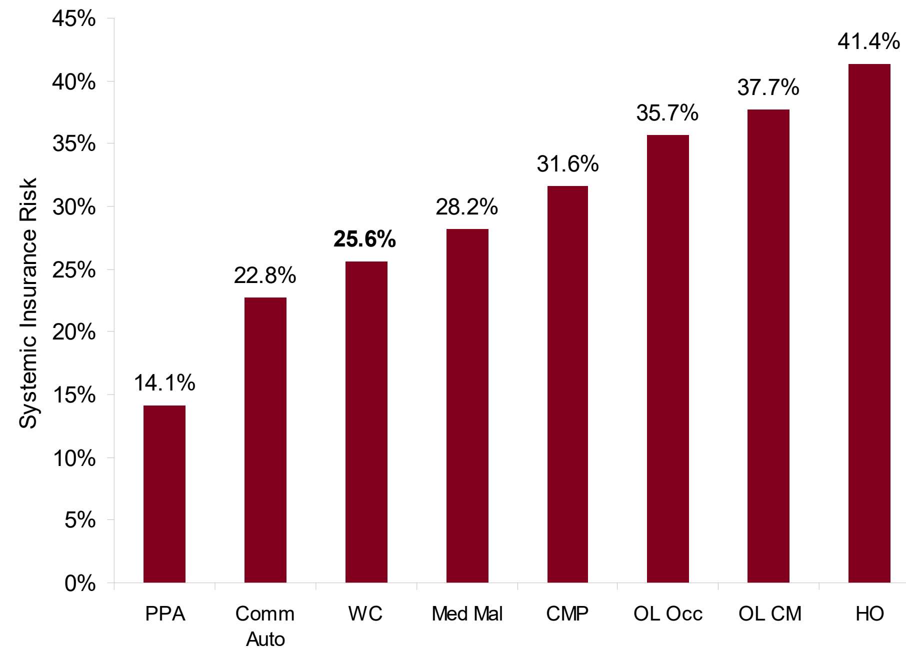
Let M_D be the moment generating function of D, for D=A, C, N or X. Let x=E(X), n=E(N), a=E(A)=nx. Then

$$\begin{aligned}
\lim_{n\to\infty} M_{A/a}(t) &= \lim_{n\to\infty} M_A(t/a) \\
&= \lim_{n\to\infty} M_C(n(M_X(t/a)-1)) \\
&= \lim_{n\to\infty} M_G(n(M'_X(0)t/nx + R(t/nx))) \\
&= \lim_{n\to\infty} M_C(t + nR(t/nx)) \\
&= M_C(t)
\end{aligned}$$

For some remainder function $R(t)=O(t^2)$. The assumptions on X guarantee that $M_X'(0)=x=E(X)$ & that the reminder term in Taylor's expansion is $O(t^2)$. The result follows because a distribution is uniquely determined by its moment generating function.

Systemic Insurance Risk by Line

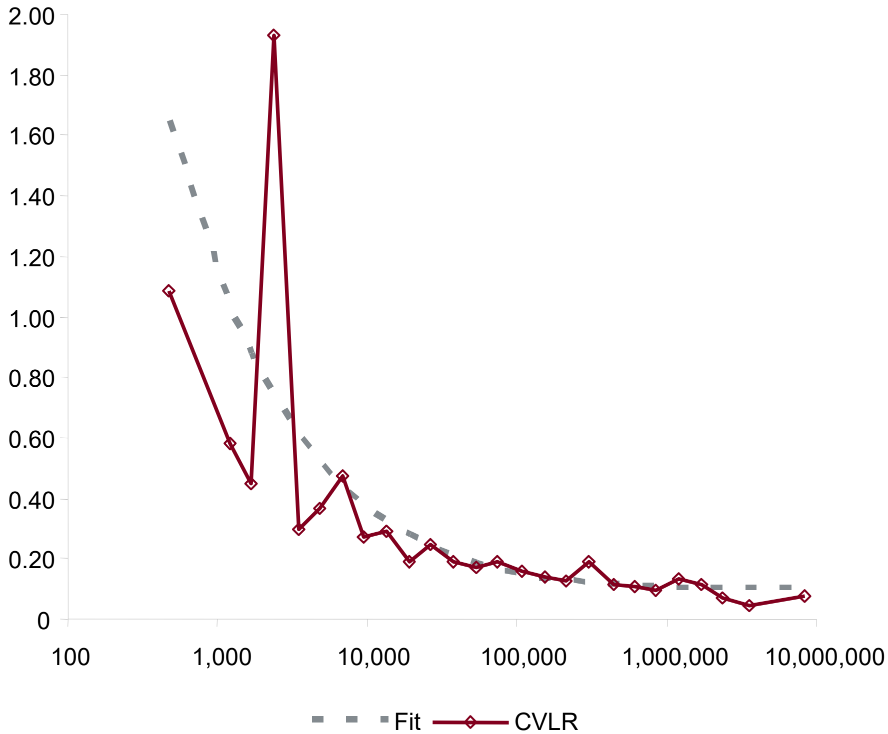


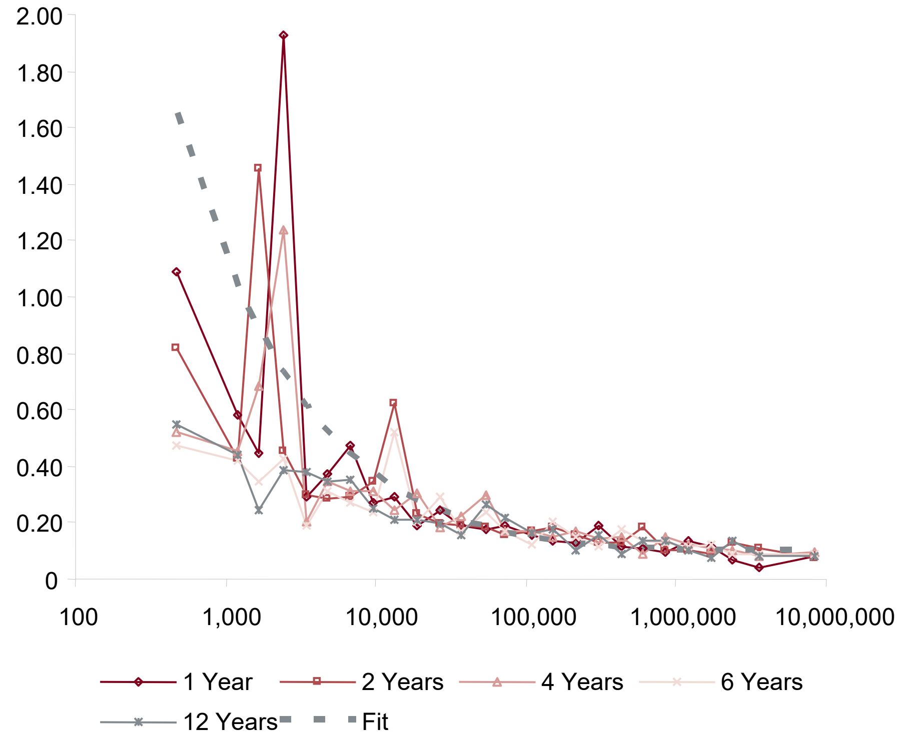


- Systemic risk quantified using study of Schedule P gross ultimate loss ratios, cf. Slide 9
- Systemic insurance risk includes line of business uncertainty caused by

Pricing cycle	Loss reserve uncertainty
Frequency & severity trend	Legal & judicial changes
Economic activity	Weather

Volumetric/Temporal Symmetry





- Consider volatility of A(x,t), A(2x,t/2), A(4x,t/4) etc.
- Same relationship between volatility and volume, xt
- Consistent with volumetric/temporal symmetry

6. Four Levy Process Models

- A(x,t) = X(xt)
- A(x,t) = X(xZ(t))
- A(x,t) = X(xCt)
- A(x,t) = X(xCZ(t))
- A(x,t) = xR(t)

Table 2: Variance of IM1-4 and AM

Model	Variance	$v(x,t)$	Diversifying $x \to \infty$	Diversifying $t \to \infty$
$X(xt)$	$\sigma^2 xt$	$\frac{\sigma}{\sqrt{xt}}$	Yes	Yes
$X(xZ(t))$	$xt(\sigma^2 + x\tau^2)$	$\sqrt{\frac{\sigma^2}{xt} + \frac{\tau^2}{t}}$	No	Yes
$X(xCt)$	$xt(\sigma^2 + cxt)$	$\sqrt{\frac{\sigma^2}{xt} + c}$	No	No
$X(xCZ(t))$	$x^2t^2\left(\frac{(c+1)\tau^2}{t} + c\right) + \sigma^2 xt$	$\sqrt{\frac{\sigma^2}{xt} + \frac{\tau'^2}{t} + c}$	No	No
$xX(t)$	$x^2\sigma^2 t$	$\sigma/\sqrt{t}$	Const.	Yes

$\tau' = (1+c)\tau$

6. Four Levy Process Models

- Which model is consistent with the data?

• $A(x,t) = X(xt)$	no	Volumetrically diversifying
• $A(x,t) = X(xZ(t))$	no	Volumetric/temporal asymmetry
• $A(x,t) = X(xCt)$	Yes	
• $A(x,t) = X(xCZ(t))$	no	Volumetric/temporal asymmetry
• $A(x,t) = xR(t)$	no	Constant volatility with volume

Directions and Credibility

- Levy process defines direction through jump distribution (modulo drift, Brownian component)
  - Frequency mixing, C or Z, corresponds to speed along a defined direction
  - Severity mixing corresponds to different direction

7. Why bother with Levy Processes?

- Paper uses compound Poisson distributions as examples for simplicity

- Why bother with general Levy processes?
  - Publishing cottage industry!

- “Infinite activity” Levy processes include processes with X(1) distributed as
  - Lognormal
  - Pareto
  - Gamma
  - Laplace
  - Weibull ($\alpha<1$; $\alpha>1$ is not infinitely divisible)

8. So What? Can we see the impact in prices?

- Idiosyncratic risk matters, price should decrease with size
  - Price = margin or spread over actuarial rate
  - Size = expected loss = xt; t=1
  - Large depends on particulars of severity distribution
- Umbrella and high limit policies
  - Companies target higher price for higher process risk
- Reinsurer notion of “balance”
  - Unbalanced cover has premium < limit
- Large accounts, package policies
  - Probably top-line focus rather than risk theory
- Myers-Cohn